# Renee on the Mandell-May def G-spectra

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## Motivation

classical spectrum $E$ is $\{E_n, \Sigma E_n \to E_{n+1} : n \geq 0\}$

rephrase Let $\mathbb{N}$ be discrete cat whose objects $n \in \mathbb{N}$
symmetric monoidal under addition

Consider $S_{\mathbb{N}} \in \mathrm{Fun}(\mathbb{N}, Top_*)$ given by $n \mapsto S^n$

The functor category is symmetric monoidal and $S_{\mathbb{N}}$ is a monoid, but is not commutative

<u>Def</u> A spectrum $E$ is a functor $E : \mathbb{N} \to Top_*$ that is a module over $S_{\mathbb{N}}$. These modules do <u>not</u> form a SMC since $S_{\mathbb{N}}$ is not comm.

$S_{\mathbb{N}}$ is not comm because

(1)

$$\begin{tikzcd} S_{\mathbb{N}} \otimes S_{\mathbb{N}} \arrow[dr, "\mu"] \arrow[d, "\tau"'] & \\ S_{\mathbb{N}} \otimes S_{\mathbb{N}} \arrow[r, "\mu"'] & S_{\mathbb{N}} \end{tikzcd} \qquad \begin{tikzcd} S^m \wedge S^n \arrow[dr] \arrow[d, "\tau"'] & \\ S^n \wedge S^m \arrow[r] & S^{m+n} \end{tikzcd}$$

does not commute but does up to orthogonal transformation

A functor $E : \mathbb{N} \to Top_*$ does not encode the usual structure of a specture, but the module structure over $S_{\mathbb{N}}$ does.

$$\begin{tikzcd} E_n \wedge S^k \wedge S^l \arrow[r] \arrow[d] & E_n \wedge S^{k+l} \arrow[d] \\ E_{n+k} \wedge S^l \arrow[r] & E_{n+k+l} \end{tikzcd}$$

commutes

so the module structure is associative.

We need a better indexing category than $\mathbb{N}$.
Let $\mathcal{O}$ be cat with $ob(\mathcal{O}) = \mathbb{N}$ and

$\mathcal{O}(n,n) = O(n)$ = orthogonal gp.

There are no morphisms $m \to n$ for $m \neq n$.

$\mathcal{O}$ is the skeleton of the category of fin dim Euclidean vector spaces + orth isomorphisms

Let $S_{\mathcal{O}} \in Fun(\mathcal{O}, Top_*)$ $\quad n \mapsto S^n$

This functor category also has a Day SM structure. The analog of $\mathcal{O}$ commutes (?). Day convolution is a Kan extension

$$\begin{tikzcd}
\mathcal{O} \times \mathcal{O} \arrow[r, "E \times F"] \arrow[d, "+"'] & Top_* \times Top_* \arrow[r, "\wedge"] & Top_* \\
\mathcal{O} \arrow[urr, dashed, "E \wedge F"'] & &
\end{tikzcd}$$

<u>Def</u> An <u>orthogonal spectrum</u> is a functor $E : \mathcal{O} \to Top_*$ that is a module $/ S_{\mathcal{O}}$.

This means $E_n$ has an $O(n)$ action.

<u>We want to internalize the module structure into the indexing category</u>

<u>Def</u> Let $\mathcal{J}$ be the category whose objects are Euclidean vector spaces. Let $\mathcal{O}(V,W)$ = Stiefel mfd of orth ...

embeddings $V \to W$.
For each $\varphi \in \mathcal{O}(V, W)$ let $W - \varphi(V)$ be orth complement

This defines a vector bundle on $O(V,W)$
Then the morphism space $J(V,W)$ is its Thom space.

ex $J(V,V) = O(V)_+$

$J$ is symm monoidal under $\oplus$

$$J(V,W) \wedge J(V',W') \longrightarrow J(V \oplus V', W \oplus W')$$

$J$ is enriched over $Top_*$. Composition.

$$J(V,W) \wedge J(U,V) \longrightarrow J(U,W)$$

<u>Redef</u> An <u>orthogonal spectrum</u> is a functor $J \xrightarrow{E} Top_*$, $V \longmapsto E_V$

$$J(V,W) \longrightarrow Sp(E_V, E_W)$$

leads via some adjunction

$$J(V,W) \wedge E_V \longrightarrow E_W$$

e.g. $J(\mathbb{R}^n, \mathbb{R}^{n+1}) = Thom(\mathbb{R}^1 \longrightarrow E \longrightarrow O(n,n+1))$

$$= \text{“} \bigvee_{O(n,n+1)} S^1 \text{”}$$

$$J(\mathbb{R}^n, \mathbb{R}^{n+1}) \wedge E_n = \bigvee_{O(n,n+1)} S^1 \wedge E_n \longrightarrow E_{n+1}$$

The sphere spectrum and Yoneda spectrum

$S^{-0} : V \longrightarrow S^V = \mathcal{J}(0, V)$

More generally $S^{-V} : W \to \mathcal{J}(V, W)$

Yoneda spectrum. These

$\mathcal{J}^{op} \longrightarrow \mathrm{Fun}(\mathcal{J}, Top_*)$ $\quad Sp := \mathrm{Fun}(\mathcal{J}, Top_*)$

$V \longrightarrow \mathcal{J}(V, -)$

Enriched Yoneda lemma gives

$\mathrm{Hom}_{Sp}(S^{-V}, E) = E_V$

Enter a finite gp $G$: There are two cats of pointed $G$-spaces, using equiv on all cont maps.

$\mathcal{T}^G$ = cat of pointed $G$-spaces and $G$-maps

$\mathcal{T}_G$ = " and cont map

$\mathcal{T}^G(X, Y)$ is in $Top_*$

$\mathcal{T}_G(X, Y)$ is in $\mathcal{T}^G$

Def The Mandell-May category $\mathcal{J}_G$ has as objects orth reps $V$ of $G$.

$\mathcal{J}_G(V, W)$ is $\mathcal{J}(V, W)$ as before with a $G$-action.

Def A $G$-spectrum is a functor

$\mathcal{J}_G \xrightarrow{E} \mathcal{T}_G$

$V \longmapsto E_V$

Naive versus genuine spectra

$\mathcal{J} \longrightarrow \mathcal{J}_G$ sub cat of Vector spaces with trivial $G$-action.

$\mathcal{J} \longrightarrow \mathcal{J}_G \xrightarrow{E} \mathcal{T}_G$ Such a functor is a <u>naive $G$-spectrum</u>.

A functor from $\mathcal{J}_G$ is determined by its value on $\mathcal{J}$. The structure map factors

$$\begin{tikzcd} \mathcal{J}_G(V,W) \wedge E_V \arrow[rr, "\varepsilon_{V,W}"] \arrow[dr] & & E_W \\ & \mathcal{J}_G(V,W) \wedge_{O(V)} E_V \arrow[ur] & \end{tikzcd}$$

If $|V|=|W|$ then $E_V \approx E_W$.

The categories of naive + genuine spectra are equivalent. $\mathcal{J}$ and $\mathcal{J}_G$ are equivalent. However $Sp^G$ and $Sp^G_{naive}$ have different MC structures

Given a spectrum $E$ and a space $X$ we define $E \wedge X$ by $(E \wedge X)_V = E_V \wedge X$.

We denote $E \wedge S^W$ by $\Sigma^W E$

Define $F_G(X,E)$ by $F_G(X,E)_V = F_G(X, E_V)$

$F_G(S^W, E) =: \Omega^W E$

Tautological presentation

Any spectrum $E$ is the reflexive coequalizer (i.e. colim over $(\rightrightarrows)$) of

$$\bigvee_{V,W} S^{-W}\wedge \mathcal{J}_G(V,W)\wedge E_V \;\substack{\xrightarrow{\mathcal{J}_{V,W}\wedge E_V} \\ \xleftarrow{\text{bonait}} \\ \xrightarrow[S^{-W}\wedge \mathcal{E}_{V,W}]{}}\; \bigvee_V S^{-V}\wedge E \longrightarrow E$$

Abbreviate this as $\operatorname{colim}_V S^{-V}\wedge E_V$

<u>Smash product of spectra</u>

$$\begin{tikzcd}
\mathcal{J}_G\times\mathcal{J}_G \arrow[r, "E\times F"] \arrow[d, "\oplus"'] & \mathcal{T}_G\times\mathcal{T}_G \arrow[r, "\wedge"] \arrow[dl, phantom, "\Downarrow"] & \mathcal{T}_G \\
\mathcal{J}_G \arrow[urr, dashed] & &
\end{tikzcd}$$

Abbreviate to

$$E\wedge F = \operatorname*{colim}_{V,W} S^{-V\oplus W}\wedge E_V\wedge F_W$$

i.e. reflexive coequalizer of

$$\bigvee_{V,V',W,W'} S^{-W\oplus W'}\wedge \mathcal{J}(V,W)\wedge \mathcal{J}(V',W')\wedge E_V\wedge F_{V'}$$

$$\downarrow\ \uparrow\ \downarrow$$

$$\bigvee_{V,V'} S^{-V\oplus V'}\wedge E_V\wedge F_{V'}$$